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Laurent Continued Fractions Corresponding to Pairs of Power Series*

OLAV NJÅSTAD

Department of Mathematics, University of Trondheim-NTH, N-7034 Trondheim, Norway

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A continued fraction $\sum_{k=1}^{\infty} (\alpha_k(z)/\beta_k(z))$ is said to correspond to the power series $\sum_{p=0}^{\infty} c_p z^{-p}$ and $\sum_{p=1}^{\infty} -c_{-p} z^p$ if series expansions of the following form for the approximants $f_k(z)$ of the continued fraction are valid,

$$f_k(z) - \sum_{p=0}^{\mu_k} c_p z^{-p} = cz^{-(\mu_k+1)} + \cdots,$$

$$f_k(z) + \sum_{p=1}^{\nu_k} c_{-p} z^p = dz^{\nu_k+1} + \cdots,$$

where $\mu_k, \nu_k \to \infty$ when $k \to 0$. We introduce a special class of continued fractions, Laurent fractions, and show that the concept of correspondence above induces a one-to-one mapping between all Laurent fractions and all double sequences $\{c_n: n = 0, \pm 1, \pm 2, ...\}$ of real numbers satisfying the determinant conditions $H_{2m}^{(-2m)} \neq 0$, $H_{2m+1}^{(-2m)} \neq 0$, $m = 0, 1, 2...$. ($H_q^{(p)}$ are the Hankel determinants associated with the sequence $\{c_n\}$.) A subclass, the contractive Laurent fractions, is mapped onto those double sequences which satisfy the conditions $H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$, $m = 0, 1, 2...$. The double sequences, which in addition to $H_{2m}^{(2m)} \neq 0$, $H_{2m+1}^{(-2m)} \neq 0$ also satisfy the conditions $H_{2m-1}^{(-2m+1)} \neq 0$, $H_{2m}^{(-2m+1)} \neq 0$, $m = 0, 1, 2, ...$, are those associated with general T-fractions (or M-fractions).

INTRODUCTION

Let $\{c_n: n = 0, 1, 2, ...\}$ be a sequence of real numbers. The *Hankel determinants* $H_k^{(n)}$ are defined for $n = 0, 1, 2, ...$ as follows,

$$H_0^{(n)} = 1, \qquad H_k^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_{n+1} & & & \vdots \\ \vdots & & & \vdots \\ c_{n+k-1} & \cdots & \cdots & c_{n+2k-2} \end{vmatrix} \qquad \text{for} \quad k = 1, 2, 3, ...$$

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When $\{c_n : n=0, \pm 1, \pm 2, ...\}$ is a double sequence, the Hankel determinants are defined as above for $n=0, \pm 1, \pm 2, ...$.

With a given (simple) sequence $\{c_n\}$ we associate the formal power series $\sum_{k=0}^{\infty} c_k z^{-k}$, and with a given double sequence $\{c_n\}$ we associate the two formal power series $\sum_{k=0}^{\infty} c_k z^{-k}$ and $\sum_{k=1}^{\infty} -c_{-k} z^k$.

A continued fraction

$$\mathop{\mathrm{K}}_{k=1}^{\infty} \frac{\alpha_k(z)}{\beta_k(z)} = \frac{\alpha_1(z)}{\beta_1(z)+} \frac{\alpha_2(z)}{\beta_2(z)+} \frac{\alpha_3(z)}{\beta_3(z)+} \cdots$$

is said to *correspond* to the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z=\infty$ if formal power series expansions of the following form are valid (we write $f_k(z)$ for the kth approximant of the continued fraction) for every k:

$$f_k(z) - \sum_{p=0}^{\mu_k} c_p z^{-p} = c z^{-(\mu_k+1)} + \cdots,$$

where $\mu_k \to \infty$ as $k \to \infty$. The continued fraction is said to correspond to the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z=\infty$ and to the series $\sum_{k=1}^{\infty} -c_{-k} z^k$ at $z=0$ if formal power series expansions of the following form are valid,

$$f_k(z) - \sum_{p=0}^{\mu_k} c_p z^{-p} = c z^{-(\mu_k+1)} + \cdots,$$

$$f_k(z) + \sum_{p=1}^{\nu_k} c_{-k} z^{p} = c z^{(\nu_k+1)} + \cdots,$$

where $\mu_k \to \infty$, $\nu_k \to \infty$ as $k \to \infty$.

A *modified regular C-fraction* is a continued fraction of the form

$$\frac{a_1}{1} + \frac{a_2}{z} + \frac{a_3}{1} + \frac{a_4}{z} + \cdots, \qquad a_k \neq 0 \qquad \text{for} \quad k=1, 2, ...$$

It is called a *modified Stieltjes fraction* if $a_k > 0$ for $k=1, 2, ...$.

A *J-fraction* is a continued fraction of the form

$$\frac{g_1 z}{z+h_1} - \frac{g_2}{z+h_2} - \frac{g_3}{z+h_3} - \cdots, \qquad g_k \neq 0 \qquad \text{for} \quad k=1, 2, ...$$

It is called a *real J-fraction* if $g_k > 0$ for $k=1, 2, ...$.

By a *general T-fraction* we mean a continued fraction of the form

$$\frac{F_1 z}{1+G_1 z} + \frac{F_2 z}{1+G_2 z} + \frac{F_3 z}{1+G_3 z} + \cdots, \qquad F_k \neq 0,\ G_k \neq 0 \text{ for } k=1, 2, ...$$

It is called a *positive T-fraction* if $F_k > 0$, $G_k > 0$ for $k=1, 2, ...$.

For more details on these concepts see, e.g., [5].

It is well known that the concept of correspondence at $z=\infty$ induces a one-to-one mapping between all modified regular C-fractions and all (simple) sequences $\{c_n\}$ satisfying the condition

$$H_k^{(0)} \neq 0, \qquad H_k^{(1)} \neq 0, \qquad \text{for} \quad k=0, 1, 2, \ldots.$$

The modified Stieltjes fractions are mapped onto the sequences where $H_k^{(0)} > 0$ and $(-1)^k H_k^{(1)} > 0$.

More generally this concept of correspondence induces a one-to-one mapping between all J-fractions and all (simple) sequences satisfying the condition

$$H_k^{(0)} \neq 0 \qquad \text{for} \quad k=0, 1, 2, \ldots.$$

The real J-fractions are mapped onto the sequences where $H_k^{(0)} > 0$. A real J-fraction is the even part of a modified regular C-fraction iff also $H_k^{(1)} \neq 0$ in the corresponding sequence and of a modified Stieltjes fraction iff also $H_k^{(1)} > 0$. (For the concept of the even part of a continued fraction see, e.g., [5, p. 38–41].)

For more details on these correspondence results see, e.g., [1, 5, 12].

More recently it has become known that the concept of correspondence at $z=\infty$ and at $z=0$ induces a one-to-one mapping between all general T-fractions and all double sequences $\{c_n\}$ satisfying the condition $H_{2m}^{(-2m)} \neq 0$, $H_{2m+1}^{(-2m)} \neq 0$, $H_{2m}^{(-2m+1)} \neq 0$, $H_{2m-1}^{(-2m+1)} \neq 0$ for $m=0, 1, 2, \ldots$. The positive T-fractions are mapped onto the sequences where $H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$, $H_{2m}^{(-2m+1)} > 0$, $H_{2m-1}^{(-2m+1)} < 0$.

Correspondence results for general T-fractions can also be formulated in terms of the closely related *M-fractions*

$$\frac{F_1}{1+G_1 z} + \frac{F_2 z}{1+G_2 z} + \frac{F_3 z}{1+G_3 z} + \cdots$$

introduced in [7, 8]. For these correspondence results at $z=\infty$ and at $z=0$ see [4, 5, 6, 7, 8].

In this paper we show that the concept of correspondence at $z=\infty$ and at $z=0$ more generally induces a one-to-one mapping between a class of continued fractions called *Laurent fractions* (for definition see Section 1) and all double sequences $\{c_n\}$ satisfying the condition

$$H_{2m}^{(-2m)} \neq 0, \qquad H_{2m+1}^{(-2m)} \neq 0 \qquad \text{for} \quad m=0, 1, 2, \ldots.$$

A special class of Laurent fractions called *contractive Laurent fractions* (for definition see Section 1) is mapped onto the class of those sequences where

$H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$. A contractive Laurent fraction is then equivalent with a general T-fraction iff also $H_{2m}^{(-2m+1)} \neq 0$, $H_{2m-1}^{(-2m+1)} \neq 0$ in the corresponding double sequence, and with a positive T-fraction iff also $H_{2m}^{(-2m+1)} > 0$, $H_{2m-1}^{(-2m+1)} < 0$. (For the concept of equivalence of continued fractions see, e.g., [5, p. 31].) The general T-fractions where $H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$, $H_{2m}^{(-2m+1)} \neq 0$, $H_{2m-1}^{(-2m+1)} \neq 0$ are the *APT-fractions* (characterized by $F_{2m-1}F_{2m} > 0$, $F_{2m-1}G_{2m-1} > 0$) studied in [2].

The contractive Laurent fractions are connected with orthogonal Laurent polynomials (see [3, 9]). The relationship is as follows. Let $R_n(z)$ be the monic orthogonal Laurent polynomials determined by the functional Φ, where $\Phi(\sum_{i=p}^{q} r_i z^i) = \sum_{i=p}^{q} r_i c_i$. Let $B_k(z)$ be the denominators of the Laurent fraction corresponding to the series $\sum_{p=0}^{\infty} c_p z^{-p}$, $\sum_{p=1}^{\infty} -c_{-p} z^p$. Then $R_{n_k}(z) = B_k(z)$ for every non-singular index n_k. When $n_k + 1$ is singular and $n_k + 1 = 2m$, then $R_{2m}(z) = k \cdot zB_k(z)$. When $n_k + 1$ is singular and $n_k + 1 = 2m + 1$, then $R_{2m+1}(z) = k' \cdot z^{-1}B_k(z)$. (These results follows from recursion formulas in [9].)

Orthogonal Laurent polynomials can be used to solve the strong Hamburger moment problem (see [3, 9]). In a forthcoming paper it will be shown that the problem also can be solved by the use of contractive Laurent fractions (see [10]). It has earlier been shown that the problem can be solved by APT-fractions in the nonsingular case (see [2]).

The idea of generalizing M-fractions or general T-fractions to obtain results on the strong Hamburger moment problem has also been utilized in [11].

For definitions and basic properties concerning continued fractions we refer to [5].

1. Laurent Fractions

Let S be a subsequence of the sequence $N = \{0, 1, 2, 3, ...\}$ of non-negative integers, with the property that no two consecutive elements of N belong to S. We shall call the elements of S *singular* indices, and the elements of $N - S$ non-singular indices. We shall denote by T the set of all triples of consecutive non-singular indices (i.e., triples of non-singular indices where there are no non-singular indices in between).

We define for every non-singular index n an ordered pair $(a_n, b_n) = (a_n(z), b_n(z))$ (where z is an arbitrary index n an ordered number different from zero) in the following way:

L_1. For every non-singular index n there is given a real number $v_n \neq 0$, and $v_0 = 1$.

L_{II}. For every non-singular index n there is given a real number q_n, where $q_0 = 0$, $q_n \neq 0$ for $n \neq 0$.

L_{III}. For every singular index n there is a given real number w_n.

The complex numbers a_n, b_n are given as follows:

L_1. $a_{2m} = q_{2m}$, $b_{2m} = v_{2m} + (1/v_{2m-1})z$, when $(2m, 2m-1, 2m-2) \in T$.

L_2. $a_{2m} = q_{2m}z$, $b_{2m} = v_{2m} + (1/v_{2m-1})z$, when $(2m, 2m-1, 2m-3) \in T$.

L_3. $a_{2m} = q_{2m}$, $b_{2m} = (v_{2m}/v_{2m-2})z^{-1} + w_{2m-1} + z$, when $(2m, 2m-2, 2m-4) \in T$.

L_4. $a_{2m} = q_{2m}z$, $b_{2m} = (v_{2m}/v_{2m-2})z^{-1} + w_{2m-1} + z$, when $(2m, 2m-2, 2m-3) \in T$.

L_5. $a_{2m+1} = q_{2m+1}$, $b_{2m+1} = (1/v_{2m})z^{-1} + v_{2m+1}$, when $(2m+1, 2m, 2m-1) \in T$.

L_6. $a_{2m+1} = q_{2m+1}z^{-1}$, $b_{2m+1} = (1/v_{2m})z^{-1} + v_{2m+1}$, when $(2m+1, 2m, 2m-2) \in T$.

L_7. $a_{2m+1} = q_{2m+1}$, $b_{2m+1} = z^{-1} + w_{2m} + (v_{2m+1}/v_{2m-1})z$, when $(2m+1, 2m-1, 2m-3) \in T$.

L_8. $a_{2m+1} = q_{2m+1}z^{-1}$, $b_{2m+1} = z^{-1} + w_{2m} + (v_{2m+1}/v_{2m-1})z$, when $(2m+1, 2m-1, 2m-2) \in T$.

(We consider $n = -1$ as a non-singular index in these formulas).

Let $\{n_k : k = 0, 1, 2, \ldots\} = N - S$ be the sequence of non-singular indices. We shall write $\alpha_k = \alpha_k(z) = a_{n_k}(z)$, $\beta_k = \beta_k(z) = b_{n_k}(z)$ for $k = 1, 2, 3, \ldots$. We note that $\alpha_k \neq 0$ for every k. Therefore $\{(\alpha_k, \beta_k) : k = 1, 2 \ldots\}$ is the sequence of elements of a continued fraction $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$. A continued fraction obtained in this way is called a *Laurent fraction*. We shall call a Laurent fraction *non-singular* if all indices are non-singular.

Let $A_k(z)$ and $B_k(z)$ denote the numerator and denominator of the kth approximant $f_k(z)$ of this continued fraction. Then $A_k(z)$ and $B_k(z)$ satisfy the following recursion formulas:

$$\begin{aligned} A_k &= \beta_k A_{k-1} + \alpha_k A_{k-2} \quad \text{for} \quad k = 1, 2, \ldots, A_{-1} = 1, A_0 = 0, \\ B_k &= \beta_k B_{k-1} + \alpha_k B_{k-2} \quad \text{for} \quad k = 1, 2, \ldots, B_{-1} = 0, B_0 = 1. \end{aligned} \tag{1.1}$$

We note that $A_1 = q_1$, $B_1 = z^{-1} + v_1$ if $n_1 = 1$ (in view of L_5 and R), while $A_1 = q_2 z$, $B_1 = v_2 z^{-1} + w_1 + z$ if $n_1 = 2$ (in view of L_4 and R).

Theorem 1. *The functions A_k and B_k are of the form*

$$A_k(z) \quad \sum_{i=-(m-1)}^{m} a_{2m,i} z^i,$$

$$B_k(z) = \sum_{i=-m}^{m} b_{2m,i} z^i, \qquad b_{2m,m} = 1,\ b_{2m,-m} = v_{2m}, \text{ when } n_k = 2m,$$

$$A_k(z) = \sum_{i=-m}^{m} a_{2m+1,i} z^i,$$

$$B_k(z) = \sum_{i=-(m+1)}^{m} b_{2m+1,i} z^i, \qquad b_{2m+1,-(m+1)} = 1,\ b_{2m+1,m} = v_{2m+1},$$

$$\text{when } n_k = 2m+1.$$

Proof. The result follows by induction from L_1–L_8 and R. ▮

We note that we may write $A_k(z) = \Pi_{2m-1}(z)/z^{m-1}$, $B_k(z) = \Pi_{2m}(z)/z^m$ when $n_k = 2m$, $A_k(z) = \Pi_{2m}(z)/z^m$, $B_k(z) = \Pi_{2m+1}(z)/z^{m+1}$ when $n_k = 2m+1$. Here Π_r is a polynomial of degree at most equal to r.

A Laurent fraction shall be called *contractive* (because of mapping properties of the associated linear fractional transformations) when the following extra conditions are satisfied:

C_0.	$v_{n-1} \cdot v_{n+1} < 0$	when n is singular,
C_1.	$q_{2m} \cdot v_{2m} \cdot v_{2m-1} > 0$	when $(2m, 2m-1, 2m-2) \in T$,
C_2.	$q_{2m} \cdot v_{2m} < 0$	when $(2m, 2m-1, 2m-3) \in T$,
C_3.	$q_{2m} < 0$	when $(2m, 2m-2, 2m-4) \in T$,
C_4.	$q_{2m} \cdot v_{2m} < 0$	when $(2m, 2m-2, 2m-3) \in T$,
C_5.	$q_{2m+1} \cdot v_{2m+1} \cdot v_{2m} > 0$	when $(2m+1, 2m, 2m-1) \in T$,
C_6.	$q_{2m+1} \cdot v_{2m+1} < 0$	when $(2m+1, 2m, 2m-2) \in T$,
C_7.	$q_{2m+1} < 0$	when $(2m+1, 2m-1, 2m-3) \in T$,
C_8.	$q_{2m+1} \cdot v_{2m+1} < 0$	when $(2m+1, 2m-1, 2m-2) \in T$.

2. Connection with General *T*-Fractions

By a *general T-fraction* we shall here mean an continued fraction

$$\mathop{\mathrm{K}}_{k=1}^{\infty} \frac{F_k z}{1 + G_k z}, \qquad \text{where} \quad F_k \neq 0,\ G_k \neq 0,\ k = 1, 2, 3, \ldots . \tag{2.1}$$

(Note that in [4, 5] the condition $G_k \neq 0$ is not included in the definition of a general T-fraction.)

By an APT-*fraction* we mean a general T-fraction where

$$F_{2m-1}F_{2m} > 0, \qquad F_{2m-1}G_{2m-1} > 0, \qquad m = 1, 2, \ldots . \tag{2.2}$$

THEOREM 2. *There is a one-to-one correspondence between non-singular Laurent fractions and equivalent general T-fractions, given by the formulas*

$$\begin{aligned} F_{2m} &= q_{2m}\frac{v_{2m-2}}{v_{2m}}, \qquad & F_{2m+1} &= q_{2m+1}, \\ G_{2m} &= \frac{1}{v_{2m}\cdot v_{2m-1}}, \qquad & G_{2m+1} &= v_{2m}\cdot v_{2m+1} \end{aligned} \tag{2.3}$$

or inversely

$$\begin{aligned} q_{2m} &= \frac{F_{2m}}{G_{2m-1}G_{2m}}, \qquad & q_{2m+1} &= F_{2m+1}, \\ v_{2m} &= \frac{1}{G_1\cdots G_{2m}}, \qquad & v_{2m+1} &= G_1\cdots G_{2m+1}. \end{aligned} \tag{2.4}$$

The contractive non-singular Laurent fractions correspond exactly to the APT-fractions.

Proof. Let $D_n(z)$ and $E_n(z)$ be the numerators and denominators of the general T-fraction $\mathsf{K}_{n=1}^{\infty}(F_n\cdot z)/(1+G_n\cdot z)$. Then $D_n(z)$ and $E_n(z)$ satisfy the recursion formulas

$$\begin{aligned} D_n(z) &= (1+G_nz)\,D_{n-1}(z)+F_nzD_{n-2}(z), \qquad & n&=1, 2, \ldots, D_{-1}=1,\ D_1=0, \\ E_n(z) &= (1+G_nz)\,E_{n-1}(z)+F_nzE_{n-2}(z), \qquad & n&=1, \ldots, E_{-1}=0, \quad E_0=1. \end{aligned} \tag{2.5}$$

We obtain a non-singular Laurent fraction $\mathsf{K}_{n=1}^{\infty}(a_n(z)/b_n(z))$ by defining $a_n(z)$ and $b_n(z)$ through (2.4), L_1, and L_5. We observe that the functions $A_{2m}(z) = v_{2m}z^{-m}D_{2m}(z)$, $B_{2m}(z) = v_{2m}z^{-m}E_{2m}(z)$, $A_{2m+1}(z) = z^{-(m+1)}$ $D_{2m+1}(z)$, $B_{2m+1}(z) = z^{-(m+1)}E_{2m+1}(z)$ satisfy the recursion formulas (1.1). Hence $A_n(z)$ and $B_n(z)$ are numerators and denominators of $\mathsf{K}_{n=1}^{\infty}(a_n(z)/b_n(z))$. Obviously $A_n(z)/B_n(z) = D_n(z)/E_n(z)$, and consequently the two continued fractions are equivalent. Conversely for a given non-singular Laurent fraction an equivalent general T-fraction can be obtained by defining F_n and G_n by (2.3). Clearly the correspondence is one-to-one.

From (2.3) is obtained

$$F_{2m-1}G_{2m-1}=q_{2m}v_{2m-1}v_{2m-2},$$

$$F_{2m-1}F_{2m}=q_{2m-1}v_{2m-1}v_{2m-2}\cdot\frac{q_{2m}^2}{q_{2m}v_{2m}v_{2m-1}},$$

and conversely from (2.4) is obtained

$$q_{2m+1}v_{2m+1}v_{2m}=F_{2m+1}G_{2m+1},$$

$$q_{2m}v_{2m}v_{2m-1}=F_{2m-1}F_{2m}\cdot\frac{1}{F_{2m-1}G_{2m-1}G_{2m}^2}.$$

It follows that conditions C_1 and C_5 are satisfied for all m if and only if condition (2.2) is satisfied for all m. There are no non-singular indices in this situation, so C_0 is always satisfied. ∎

3. Laurent Series Corresponding to Given Laurent Fractions

We shall define correspondence between a Laurent fraction $\mathsf{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ and two series, namely with the series $\sum_{p=1}^{\infty} c_p z^{-p}$ at infinity and with the series $\sum_{p=1}^{\infty} -c_{-p}z^p$ at the origin. Equivalently we shall talk of correspondence with a *formal* Laurent series $(\sum_{p=0}^{\infty} c_p z^{-p}+\sum_{p=1}^{\infty} -c_{-p}z^p)$. Formal Laurent series are called simply Laurent series in the following. We denote the coefficients of z^p for $p=1,2,\ldots$ by $-c_{-p}$ in order to get conditions that are easily stated in terms of Hankel determinants.

We say that the Laurent fraction $\mathsf{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ corresponds to the Laurent series $\sum_{p=0}^{\infty} c_p z^{-p}+\sum_{p=1}^{\infty} -c_{-p}z^p$ if (for every k) formal power series expansions of the following forms are valid (we write $f_k(z)$ for $A_k(z)/B_k(z)$),

$$f_k(z)+[c_{-1}z+\cdots+c_{-\nu_k}z^{\nu_k}]=cz^{\nu_k+1}+\cdots,$$

$$f_k(z)-[c_0+c_1z^{-1}+\cdots+c_{\mu_k}z^{-\mu_k}]=cz^{-(\mu_k+1)}+\cdots,$$

where $\nu_k\to_{k\to\infty}\infty$, $\mu_k\to_{k\to\infty}\infty$.

Lemma 1. *For every k the product $\alpha_1\cdots\alpha_{k+1}$ has the following form (where c denotes constants different from zero)*:

$$\alpha_1 \cdots \alpha_{k+1} = c \qquad \text{when} \quad n_k = 2m,\ n_{k+1} = 2m+1, \tag{3.1}$$

$$\alpha_1 \cdots \alpha_{k+1} = cz \qquad \text{when} \quad n_k = 2m,\ n_{k+1} = 2m+2, \tag{3.2}$$

$$\alpha_1 \cdots \alpha_{k+1} = c \qquad \text{when} \quad n_k = 2m+1,\ n_{k+1} = 2m+2, \tag{3.3}$$

$$\alpha_1 \cdots \alpha_{k+1} = cz^{-1} \qquad \text{when} \quad n_k = 2m+1,\ n_{k+1} = 2m+3. \tag{3.4}$$

Proof. We note that $\alpha_1 = q_1$ if $n_1 = 1$, $\alpha_1 = q_1 z$ if $n_1 = 2$. Assume that $\alpha_1 \cdots \alpha_{k+1}$ has the form stated for $k \leqslant h$. By combining this assumption with the various forms of α_{h+2} according to L_1–L_8, we obtain the desired form of $\alpha_1 \cdots \alpha_{h+2}$. So the result follows. ▮

THEOREM 3. *The Laurent fraction* $\mathrm{K}_{k=1}^{\infty} (\alpha_k(z)/\beta_k(z))$ *corresponds to a unique Laurent series* $\sum_{p=0}^{\infty} c_p z^{-p} + \sum_{p=1}^{\infty} -c_{-p} z^p$. *For each k the following formulas hold* (*where c denotes constants different from zero*):

$$f_k(z) + [c_{-1}z + \cdots + c_{-2m}z^{2m}] = cz^{2m+1} + \cdots, \qquad \text{when} \quad n_k = 2m, \tag{3.5}$$

$$f_k(z) + [c_{-1}z + \cdots + c_{-(2m+1)}z^{2m+1}] = cz^{2m+2} + \cdots, \qquad \text{when} \quad n_k = 2m,\ n_{k+1} = 2m+2, \tag{3.6}$$

$$f_k(z) + [c_{-1}z + \cdots + c_{-(2m+1)}z^{2m+1}] = cz^{2m+2} + \cdots, \qquad \text{when} \quad n_k = 2m+1, \tag{3.7}$$

$$f_k(z) - [c_0 + \cdots + c_{2m-1}z^{-(2m-1)}] = cz^{-2m} + \cdots, \qquad \text{when} \quad n_k = 2m, \tag{3.8}$$

$$f_k(z) - [c_0 + \cdots + c_{2m}z^{-2m}] = cz^{-(2m+1)} + \cdots, \qquad \text{when} \quad n_k = 2m+1, \tag{3.9}$$

$$f_k(z) - [c_0 + \cdots + c_{2m+1}z^{-(2m+1)}] = cz^{-(2m+2)} + \cdots, \qquad \text{when} \quad n_k = 2m+1, n_{k+1} = 2m+3. \tag{3.10}$$

Proof. We write $\Delta(z)$ for the expression $A_{k+1}(z)/B_{k+1}(z) - A_k(z)/B_k(z)$. The well-known determinant formula for continued fractions (see, e.g., [5]) gives

$$\Delta(z) = (-1)^k \frac{\alpha_1(z) \cdots \alpha_{k+1}(z)}{B_k(z)\, B_{k+1}(z)}. \tag{3.11}$$

Taking into account the form of $B_k(z)$ given in Theorem 1 and the results of Lemma 1, we obtain

$\Delta(z)=cz^{2m+1}+\cdots,$ when $n_k=2m, n_{k+1}=2m+1,$

$\Delta(z)=cz^{2m+2}+\cdots,$ when $n_k=2m, n_{k+1}=2m+2,$
when $n_k=2m+1, n_{k+1}=2m+2,$ and
when $n_k=2m+1, n_{k+1}=2m+3,$

$\Delta(z)=cz^{-2m}+\cdots,$ when $n_k=2m, n_{k+1}=2m+1,$ and
when $n_k=2m, n_{k+1}=2m+2,$

$\Delta(z)=cz^{-(2m+1)}+\cdots,$ when $n_k=2m+1, n_{k+1}=2m+2,$

$\Delta(z)=cz^{-(2m+2)}+\cdots,$ when $n_k=2m+1, n_{k+1}=2m+3.$

We define c_p and $-c_p$ as the coefficients of the appropriate power series expansions for $A_k(z)/B_k(z)$ for sufficiently large k. Then the formulas of Theorem 3, and hence the desired correspondence, follow. The uniqueness of the correesponding series is immediate. ∎

4. Hankel Determinants of the Cooresponding Laurent Series

We shall develop conditions for a Laurent series to correspond to a Laurent fraction. In this section we assume that the fraction $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ corresponds to the series $\sum_{p=0}^{\infty} c_p z^{-p}+\sum_{p=1}^{\infty} -c_{-p}z^p$. Let notations be as before. We write π_k for the product $(-1)^k\alpha_1(z)\cdots\alpha_k(z)$ and note that $\pi_k\neq 0$ for $k=1, 2, \ldots$. We recall that $b_{2m,-m}\neq 0$, $b_{2m+1,m}\neq 0$ (by Theorem 1). We shall in the following simplify the notations for the Hankel determinants $H_q^{(p)}$ (see Introduction) and write H_q^p for $H_q^{(p)}$. We shall have occasion to use repeatedly the Jacobi identity

$$(H_q^p)^2-H_q^{p-1}H_q^{p+1}+H_{q+1}^{p-1}H_{q-1}^{p+1}=0 \tag{4.1}$$

(see, e.g., [5]).

For reference we list a few special cases:

$$(H_{2m+1}^{-(2m+1)})^2-H_{2m+1}^{-(2m+2)}H_{2m+1}^{-2m}+H_{2m+2}^{-(2m+2)}H_{2m}^{-2m}=0, \tag{4.2}$$

$$(H_{2m}^{-(2m-1)})^2-H_{2m}^{-2m}H_{2m}^{-(2m-2)}+H_{2m+1}^{-2m}H_{2m-1}^{-(2m-2)}=0, \tag{4.3}$$

$$(H_{2m}^{-2m})^2-H_{2m}^{-(2m+1)}H_{2m}^{-(2m-1)}+H_{2m+1}^{-(2m+1)}H_{2m-1}^{-(2m-1)}=0, \tag{4.4}$$

$$(H_{2m+1}^{-2m})^2-H_{2m+1}^{-(2m+1)}H_{2m-1}^{-(2m-1)}+H_{2m+2}^{-(2m+1)}H_{2m}^{-(2m-1)}=0. \tag{4.5}$$

Proposition 1. *Let* $n_k=2m$, $n_{k+1}=2m+1$. *Then* $H_{2m+1}^{-(2m+1)}=\pi_{k+1}H_{2m}^{-2m}$, $H_{2m+1}^{-2m}=-(\pi_{k+1}/(b_{2m,-m}\cdot b_{2m+1,m}))\,H_{2m}^{-(2m-1)}$.

Proof. The determinants formula (3.11) implies that $A_k(z)/B_k(z) = A_{k+1}(z)/B_{k+1}(z) + \pi_{k+1}/(B_k(z)\,B_{k+1}(z))$, hence by Theorem 3 we obtain

$$A_k(z) = -B_k(z)[c_{-1}z + \cdots + c_{-(2m+1)}z^{2m+1}] + \pi_{k+1}z^{m+1} + \cdots, \tag{4.6}$$

$$A_k(z) = B_k(z)[c_0 + \cdots + c_{2m}z^{-2m}] + \pi_{k+1}b_{2m+1,m}z^{-m} + \cdots, \tag{4.7}$$

where π_{k+1} by (3.1) (Lemma 1) is a constant.

Comparison of coefficients for $z^{-(m-1)}, ..., z^m$ in (4.6) gives

$$\begin{aligned} c_{-1}b_{2m,-m} \qquad\qquad\qquad &= -a_{2m,-(m-1)}, \\ \vdots \qquad\qquad\qquad\qquad &\quad\ \vdots \\ c_{-2m}b_{2m,-m} + \cdots + c_{-1}b_{2m,m-1} &= -a_{2m,m} \end{aligned} \tag{4.8}$$

Similarly comparison of coefficients for $z^{-(m-1)}, ..., z^m$ in (4.7) gives

$$\begin{aligned} c_0 b_{2m,-(m-1)} + \cdots + c_{2m-1}\cdot 1 &= a_{2m,-(m-1)}, \\ \vdots \quad &\quad \vdots \\ c_0 \cdot 1 &= a_{2m,m}. \end{aligned} \tag{4.9}$$

Addition of these equations gives

$$\begin{aligned} c_{-2m}b_{2m,-m} + \cdots + c_0\cdot 1 \qquad &= 0, \\ \vdots \qquad\qquad b \qquad\qquad &\quad \vdots \\ c_{-1}b_{2m,-m} + \cdots + c_{2m-1}\cdot 1 &= 0. \end{aligned} \tag{4.10}$$

Comparison of coefficients for z^{m+1} in (4.6) and for z^{-m} in (4.7) gives

$$c_{-(2m+1)}b_{2m,-m} + \cdots + c_{-1}\cdot 1 = \pi_{k+1}, \tag{4.11}$$

$$c_0 b_{2m,-m} + \cdots + c_{2m}\cdot 1 = -\frac{\pi_{k+1}}{b_{2m+1,m}}. \tag{4.12}$$

Straightforward calculation, applying Cramer's rule to (4.10), and using (4.11) gives

$$H_{2m+1}^{-(2m+1)} = \begin{vmatrix} c_{-(2m+1)} & \cdots & c_{-1} \\ \vdots & & \vdots \\ c_{-1} & \cdots & c_{2m-1} \end{vmatrix}$$

$$= \sum_{j=1}^{2m+1} (-1)^j c_{-j} \begin{vmatrix} c_{-2m} & \cdots & c_0 \\ \vdots & & \vdots \\ c_{-1} & \cdots & c_{2m-1} \end{vmatrix}$$

$$= -\sum_{j=1}^{2m+1} c_{-j} \begin{vmatrix} c_{-2m} & \cdots & c_0 & \cdots & c_{-1} \\ \vdots & & \vdots & & \vdots \\ c_{-1} & \cdots & c_{2m-1} & \cdots & c_{2m-2} \end{vmatrix}$$

$$= \sum_{j=1}^{2m+1} c_{-j} H_{2m}^{-2m} b_{2m,m+1-j}$$

$$= H_{2m}^{-2m} \cdot [c_{-(2m+1)} b_{2m,-m} + \cdots + c_{-1} \cdot 1] = \pi_{k+1} H_{2m}^{-2m}.$$

Similarly by applying Cramer's rule to (4.10) and using (4.12) we obtain

$$H_{2m+1}^{-2m} = \begin{vmatrix} c^{-2m} & \cdots & c_0 \\ \vdots & & \vdots \\ c_0 & \cdots & c_{2m} \end{vmatrix}$$

$$= \sum_{j=0}^{2m} (-1)^{j+1} c_j \begin{vmatrix} c_{-2m} & \cdots & c_0 \\ \vdots & & \vdots \\ c_{-1} & \cdots & c_{2m-1} \end{vmatrix}$$

$$= -\sum_{j=0}^{2m} c_j \begin{vmatrix} c_{-(2m-1)} & \cdots & c_{-2m} & \cdots & c_0 \\ \vdots & & \vdots & & \vdots \\ c_0 & \cdots & c_{-1} & \cdots & c_{2m-1} \end{vmatrix}$$

$$= \sum_{j=0}^{2m} \frac{c_j}{b_{2m,-m}} b_{2m,j-m} H_{2m}^{-(2m-1)}$$

$$= \frac{1}{b_{2m,-m}} H_{2m}^{-(2m-1)} [c_0 b_{2m,-m} + \cdots + c_{2m} \cdot 1]$$

$$= \frac{-\pi_{k+1}}{b_{2m,-m} b_{2m+1,m}} H_{2m}^{-(2m-1)}. \quad \blacksquare$$

PROPOSITION 2. *Let* $n_k = 2m$, $n_{k+1} = 2m+2$. *Then* $H_{2m+1}^{-2m} = -\pi_{k+1} z^{-1} H_{2m}^{-2m}, H_{2m+1}^{-(2m+2)} = -((\pi_{k+1} z^{-1})/(b_{2m,-m} \cdot b_{2m+2,-(m+1)}))\, H_{2m}^{-2m}$, $H_{2m+1}^{-(2m+1)} = 0$, $H_{2m+1}^{-(2m+2)} H_{2m+1}^{-2m} = H_{2m+2}^{-(2m+2)} H_{2m}^{-2m}$, $H_{2m+2}^{-(2m+1)} H_{2m}^{-2m-1)} = (H_{2m+1}^{-2m})^2$.

Proof. From (3.11) and Theorem 3 we obtain

$$A_k(z) = -B_k(z)[c_{-1} c + \cdots + c_{-(2m+2)} z^{2m+2}] + (\pi_{k+1} z^{-1}) \frac{z^{m+2}}{b_{2m+2,-(m+1)}}, \tag{4.13}$$

$$A_k(z) = B_k(z)[c_0 + \cdots + c_{2m+1} z^{-(2m+1)}] + (\pi_{k+1} z^{-1})\, z^{-m} + \cdots, \tag{4.14}$$

where $\pi_{k+1} z^{-1}$ is a constant by (3.2) (Lemma 1).

Comparison of coefficients for $z^{-(m-1)}, ..., z^{m+1}$ in (4.13) and (4.14) and addition gives

$$\begin{aligned} c_{-(2m+1)}b_{2m,-m}+\cdots+c_{-1}\cdot 1&=0,\\ \cdots\cdots\cdots&\cdots\\ c_{-1}b_{2m,-m}+\cdots+c_{2m-1}\cdot 1&=0, \end{aligned} \tag{4.15}$$

while comparison of coefficients for z^{m+2} in (4.13) and for z^{-m} in (4.14) gives

$$c_{-(2m+2)}b_{2m,-m}+\cdots+c_{-2}\cdot 1=\frac{\pi_{k+1}z^{-1}}{b_{2m+2,-(m+1)}}, \tag{4.16}$$

$$c_0 b_{2m,-m}+\cdots+c_{2m}\cdot 1=-\pi_{k+1}z^{-1}. \tag{4.17}$$

By applying Cramer's rule to the system (4.15) with the first row removed and using (4.17) we obtain in the same way as in the proof of Proposition 1 the equality $H_{2m+1}^{-2m}=-(\pi_{k+1}z^{-1})\,H_{2m}^{-2m}$. Similarly by applying Cramer's rule to (4.15) with the last row removed and using (4.16) we obtain $H_{2m+1}^{-(2m+2)}=-((\pi_{k+1}z^{-1})/(b_{2m,-m}\cdot b_{2m+2,-(m+1)}))\,H_{2m}^{-2m}$. Since (4.15) has a nontrivial solution we immediately conclude that $H_{2m+1}^{-(2m+1)}=0$. Finally the equalities $H_{2m+1}^{-(2m+2)}H_{2m+1}^{2m}=H_{2m+2}^{-(2m+2)}H_{2m}^{-2m}$, $H_{2m+2}^{-(2m+1)}H_{2m}^{-(2m-1)}=(H_{2m+1}^{-2m})^2$ follow from (4.2) and (4.5). ▮

PROPOSITION 3. *Let* $n_k=2m-1$, $n_{k+1}=2m$. *Then* $H_{2m}^{-(2m-1)}=\pi_{k+1}H_{2m-1}^{-(m-2)}$, $H_{2m}^{-2m}=(-\pi_{k+1}/(b_{2m-1,m-1}b_{2m,-m}))\,H_{2m-1}^{-(2m-1)}$.

Proof. From (3.11) and Theorem 3 we obtain

$$A_k(z)=-B_k(z)(c_{-1}z+\cdots+c_{-2m}z^{2m}]+\frac{\pi_{k+1}}{b_{2m,-m}}z^m+\cdots, \tag{4.18}$$

$$A_k(z)=B_k(z)[c_0+\cdots+c_{2m-1}z^{-(2m-1)}]+\pi_{k+1}z^{-m}+\cdots, \tag{4.19}$$

where π_{k+1} is a constant by (3.3) (Lemma 1).

Comparison of coefficients for $z^{-(m-1)}, ..., z^{m-1}$ in (4.18) and (4.19) and addition give

$$\begin{aligned} c_{-(2m-1)}\cdot 1+\cdots+\quad c_0\cdot b_{2m-1,m-1}&=0,\\ \vdots\qquad\qquad\qquad\vdots\qquad&\ \ \vdots\\ c_{-1}\cdot 1\quad+\cdots+c_{2m-2}\cdot b_{2m-1,m-1}&=0. \end{aligned} \tag{4.20}$$

Similarly comparison of coefficients for z^m in (4.18) and for z^{-m} in (4.19) give

$$c_{-2m}\cdot 1+\cdots+c_{-1}\cdot b_{2m,m-1}=\frac{\pi_{k+1}}{b_{2m,-m}}, \tag{4.21}$$

$$c_0\cdot 1+\cdots+c_{2m-1}, b_{2m,m-1}=\pi_{k+1}. \tag{4.22}$$

By applying (4.20) and (4.22) as in the proof of Proposition 1 we get $H_{2m}^{-2(2m-1)}=\pi_{k+1}H_{2m-1}^{-(2m-2)}$, while similarly applying (4.20) and (4.21) we get $H_{2m}^{-2m}=(-\pi_{k+1}/(b_{2m-1,m-1}b_{2m,-m}))\,H_{2m-1}^{-(2m-1)}$. ∎

PROPOSITION 4. $n_k=2m-1$, $n_{k+1}=2m+1$. *Then* $H_{2m}^{-2m}=(\pi_{k+1}z)\,H_{2m-1}^{-(2m-2)}$, $H_{2m}^{-(2m-2)}=((\pi_{k+1}z)/(b_{2m-1,m-1}\cdot b_{2m+1,m}))\,H_{2m-1}^{-(2m-1)}$, $H_{2m}^{-(2m-1)}=0$, $H_{2m}^{-2m}H_{2m}^{-(2m-2)}=H_{2m+1}^{-2m}H_{2m-1}^{-(2m-2)}$, $H_{2m+1}^{-(2m+1)}H_{2m-1}^{-(2m-1)}=(H_{2m}^{-2m})^2$.

Proof. From (3.11) and Theorem 3 we obtain

$$A_k(z)=-B_k(z)[c_{-1}z+\cdots+c_{-(2m+1)}z^{2m+1}]+(\pi_{k+1}z)\,z^m+\cdots, \tag{4.23}$$

$$A_k(z)=B_k(z)[c_0+\cdots+c_{2m}z^{-2m}]+\frac{\pi_{k+1}z}{b_{2m+1,m}}z^{-(m+1)}+\cdots, \tag{4.24}$$

where $\pi_{k+1}z$ is a constant by (3.4).

Comparison of coefficients for $z^{-m},\dots,z^{m-1}$ in (4.23) and (4.24) and addition gives

$$\begin{matrix} c_{-(2m-1)}\cdot 1+\cdots+c_0b_{2m-1,m-1}=0,\\ \vdots \qquad\qquad \vdots \qquad \vdots\\ c_0\cdot 1\quad+\cdots+c_{2m}b_{2m-1,m-1}=0, \end{matrix} \tag{4.25}$$

while comparison of coefficients for $z^{-(m+1)}$ in (4.24) and for z^m in (4.23) gives

$$c_1\cdot 1+\cdots+c_{2m}b_{2m-1,m-1}=\frac{\pi_{k+1}z}{b_{2m+1,m}}, \tag{4.26}$$

$$c_{-2m}\cdot 1+\cdots+c_{-1}b_{2m-1,m-1}=\pi_{k+1}z. \tag{4.27}$$

By applying (4.25) and (4.27) as in the proof of Proposition 3 we obtain $H_{2m}^{-2m}=(\pi_{k+1}z)\,H_{2m-1}^{-(2m-2)}$. Similarly by applying (4.25) and (4.26) we obtain $H_{2m}^{-(2m-2)}=((\pi_{k+1}z)/(b_{2m-1,m-1}b_{2m+1,m}))\,H_{2m-1}^{-(2m-1)}$. Also from (4.25) be get the equality $H_{2m}^{-(2m-1)}=0$. The equalities $H_{2m}^{-2m}H_{2m}^{-(2m-2)}=H_{2m+1}^{-2m}H_{2m-1}^{-(2m-2)}$ and $H_{2m+1}^{-(2m+1)}H_{2m-1}^{-(2m-1)}=(H_{2m}^{-2m})^2$ follow from (4.3) and (4.4). ∎

THEOREM 4. *Let the Laurent fraction* $\mathsf{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ *correspond to the Laurent series* $\sum_{p=0}^{\infty} c_p z^{-p} + \sum_{p=1}^{\infty} -c_{-p} z^p$. *Then the following statements hold*:

A. $H_{2m}^{-2m} \neq 0$, $H_{2m+1}^{-2m} \neq 0$ *for all* $m = 0, 1, 2, \ldots$.

B. $H_{2m}^{-(2m-1)} \neq 0$ *if and only if* $2m$ *is non-singular*, $H_{2m+1}^{-(2m+1)} \neq 0$ *if and only if* $2m+1$ *is non-singular.*

C. *If* $n_k = 2m$ *then*

$$B_k(z) = \frac{1}{H_{2m}^{-2m}} \begin{vmatrix} c_{-2m} & \cdots & c_{-1} & z^{-m} \\ \vdots & & \vdots & \vdots \\ c_0 & \cdots & c_{2m-1} & z^m \end{vmatrix};$$

if $n_k = 2m+1$ *then*

$$B_k(z) = \frac{-1}{H_{2m+1}^{-2m}} \begin{vmatrix} c_{-(2m+1)} & \cdots & c_{-1} & z^{-(m+1)} \\ \vdots & & \vdots & \vdots \\ c_0 & \cdots & c_{2m} & z^m \end{vmatrix}.$$

Proof. A and B. We note that $f_1(z) = q_1/(z^{-1} + v_1)$ if $n_1 = 1$, $f_1(z) = (q_2 z)/(v_2 z^{-1} + w_1 + z)$ if $n_1 = 2$ (see Section 1). It follows by Theorem 3 that $c_0 \neq 0$ and thus $H_1^0 \neq 0$. Similarly $c_{-1} \neq 0$ and thus $H_1^{-1} \neq 0$ if $n_1 = 1$, while $c_1 = 0$, $c_2 \neq 0$ and hence $H_1^{-1} = 0$, $H_2^{-2} \neq 0$, $H_2^{-1} \neq 0$ if $n_1 = 2$.

Now we assume that the statement on the Hankel determinants are true for all $k \leqslant h$, where $n_h = 2m$ or $n_h = 2m-1$.

(i) Let $n_h = 2m$, $n_{h+1} = 2m+1$. Then by assumption $H_{2m}^{-2m} \neq 0$ and $H_{2m}^{-(2m-1)} \neq 0$. It follows from Proposition 1 that $H_{2m+1}^{-2m} \neq 0$, $H_{2m+1}^{-(2m+1)} \neq 0$.

(ii) Let $n_h = 2m$, $n_{h+1} = 2m+2$. Then by assumption $H_{2m}^{-2m} \neq 0$ and $H_{2m}^{-(2m-1)} \neq 0$. It follows from Proposition 2 that $H_{2m+1}^{-2m} \neq 0$, $H_{2m+1}^{-(2m+1)} = 0$, $H_{2m+1}^{-(2m+2)} \neq 0$, hence $H_{2m+2}^{-(2m+2)} \neq 0$ and $H_{2m+2}^{-(2m+1)} \neq 0$.

(iii) Let $n_h = 2m-1$, $n_{h+1} = 2m$. Then by assumption $H_{2m-1}^{-(2m-1)} \neq 0$ and $H_{2m-1}^{-(2m-2)} \neq 0$. It follows from Proposition 3 that $H_{2m}^{-(2m-1)} \neq 0$, $H_{2m}^{-2m} \neq 0$.

(iv) Let $n_h = 2m-1$, $n_{h+1} = 2m+1$. Then by assumption $H_{2m-1}^{-(2m-1)} \neq 0$ and $H_{2m-1}^{-(2m-2)} \neq 0$. It follows from Proposition 4 that $H_{2m}^{-2m} \neq 0$, $H_{2m}^{-(2m-1)} = 0$, $H_{2m}^{-(2m-2)} \neq 0$, hence $H_{2m+1}^{-2m} \neq 0$ and $H_{2m+1}^{-(2m+1)} \neq 0$.

The statements of A and B now follow by induction.

C. Expansion of the determinants after the last column and comparison with (4.10), (4.15), (4.20), and (4.25) give the form of $B_k(z)$. ∎

5. The Laurent Fraction Expressed in Terms of the Corresponding Series

Also in this section we assume that the fraction $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ corresponds to the series $\sum_{p=0}^{\infty} c_p z^{-p} + \sum_{p=1}^{\infty} -c_{-p} z^p$. Our next task is to express the elements of the continued fraction in terms of the coefficients of the series.

From the form of $\mathrm{B}_k(z)$ given in Theorem 4 (or directly from Eqs. (4.10), (4.15), (4.20), and (4.25)) we find the following expressions for $b_{2m,-m}$, $b_{2m,2m-1}$, $b_{2m+1,m}$, $b_{2m+1,-m}$:

$$b_{2m,-m} = \frac{H_{2m}^{-(2m-1)}}{H_{2m}^{-2}}, \qquad b_{2m,m-1} = \frac{M_{2m}^{-2m}}{H_{2m}^{-2m}},$$
$$b_{2m+1,m} = -\frac{H_{2m+1}^{-(2m+1)}}{H_{2m+1}^{-2m}}, \qquad b_{2m+1,-m} = -\frac{N_{2m+1}^{-(2m+1)}}{H_{2m+1}^{-2m}} \tag{5.1}$$

where

$$M_{2m}^{-2m} = \begin{vmatrix} c_{-2m} & \cdots & c_{-1} \\ \vdots & & \vdots \\ c_{-2} & \cdots & c_{2m-3} \\ c_0 & \cdots & c_{2m-1} \end{vmatrix}, \qquad N_{2m+1}^{-(2m+1)} = \begin{vmatrix} c_{-(2m+1)} & \cdots & c_{-1} \\ c_{-(2m-1)} & \cdots & c_1 \\ \vdots & & \vdots \\ c_0 & \cdots & c_{2m} \end{vmatrix}. \tag{5.2}$$

THEOREM 5. *Let* $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ *correspond to* $\sum_{p=0}^{\infty} c_p z^{-p} + \sum_{p=1}^{\infty} -c_{-p} z^p$. *Then the coefficients* q_n *in* $\alpha_k(z)$ *be expressed as follows*:

$$q_{2m} = -\frac{H_{2m-2}^{-(2m-2)} H_{2m}^{-(2m-1)}}{H_{2m-1}^{-(2m-1)} H_{2m-1}^{-(2m-2)}} \quad \textit{in case } \mathrm{L}_1, \tag{5.3}$$

$$q_{2m} = -\frac{H_{2m}^{-(2m-1)} H_{2m-3}^{-(2m-4)}}{H_{2m-1}^{-(2m-2)} H_{2m-2}^{-(2m-2)}} \quad \textit{in case } \mathrm{L}_2, \tag{5.4}$$

$$q_{2m} = -\frac{H_{2m-1}^{-(2m-2)} H_{2m-4}^{-(2m-4)}}{H_{2m-2}^{-(2m-2)} H_{2m-3}^{-(2m-4)}} \quad \textit{in case } \mathrm{L}_3, \tag{5.5}$$

$$q_{2m} = \frac{H_{2m-1}^{-(2m-2)} H_{2m-3}^{-(2m-4)}}{H_{2m-2}^{-(2m-2)} H_{2m-2}^{-(2m-3)}} \quad \textit{in case } \mathrm{L}_4, \tag{5.6}$$

$$q_{2m+1} = -\frac{H_{2m+1}^{-(2m+1)} H_{2m-1}^{-(2m-2)}}{H_{2m}^{-2m} H_{2m}^{-(2m-1)}} \quad \textit{in case } \mathrm{L}_5, \tag{5.7}$$

$$q_{2m+1} = \frac{H_{2m+1}^{-(2m+1)} H_{2m-2}^{-(2m-2)}}{H_{2m}^{-2m} H_{2m-1}^{-(2m-2)}} \quad \textit{in case } \mathrm{L}_6, \tag{5.8}$$

$$q_{2m+1} = -\frac{H_{2m}^{-2m} H_{2m-3}^{-(2m-4)}}{H_{2m-1}^{-(2m-2)} H_{2m-2}^{-(2m-2)}} \quad \text{in case } \mathrm{L}_7, \tag{5.9}$$

$$q_{2m+1} = -\frac{H_{2m}^{-2m} H_{2m-2}^{-(2m-2)}}{H_{2m-1}^{-(2m-2)} H_{2m-1}^{-(2m-1)}} \quad \text{in case } \mathrm{L}_8. \tag{5.10}$$

Proof. In the cases L_1–L_8 the coefficient q_n can be written as (1) $q_{2m} = -\pi_{k+1}/\pi_k$, (2) $q_{2m} = -\pi_{k+1}/z\pi_k$, (3) $q_{2m} = -\pi_{k+1}/\pi_k$, (4) $q_{2m} = -z^{-1}\pi_{k+1}/\pi_k$, (5) $q_{2m+1} = -\pi_{k+1}/\pi_k$, (6) $q_{2m+1} = \pi_{k+1}/z^{-1}\pi_k$, (7) $q_{2m+1} = -\pi_{k+1}/\pi_k$, (8) $q_{2m+1} = -z\pi_{k+1}/\pi_k$. (Here $n_{k+1} = 2m$, $n_{k+1} = 2m+1$, respectively). Substitution for π_{k+1}, π_k by those expressions in the appropriate propositions in Section 4 that to not contain coefficients of B_k lead to the desired formulas. ∎

THEOREM 6. *The expressions v_{2m}, v_{2m+1} occurring in the elements $\beta_k(z)$ are given by*

$$v_{2m} = \frac{H_{2m}^{-(2m-1)}}{H_{2m}^{-2m}}, \qquad v_{2m+1} = -\frac{H_{2m+1}^{-(2m+1)}}{H_{2m+1}^{-2m}}. \tag{5.11}$$

Proof. This follows immediately from Theorem 1 and formulas B. ∎

THEOREM 7. *The expressions w_{2m}, w_{2m-1} occurring in the elements $\beta_k(z)$ are given by*

$$w_{2m-1} = \frac{M_{2m}^{-2m}}{H_{2m}^{-2m}} - \frac{M_{2m-2}^{-(2m-2)}}{H_{2m-2}^{-(2m-2)}} \quad \text{in case } \mathrm{L}_3 \tag{5.12}$$

$$w_{2m-1} = \frac{M_{2m}^{-2m}}{H_{2m}^{-2m}} - \frac{M_{2m-2}^{-(2m-2)}}{H_{2m-2}^{-(2m-2)}} + \frac{H_{2m-1}^{-(2m-2)} H_{2m-3}^{-(2m-3)}}{H_{2m-2}^{-(2m-2)} H_{2m-2}^{-(2m-3)}} \quad \text{in case } \mathrm{L}_4 \tag{5.13}$$

$$w_{2m} = -\frac{N_{2m+1}^{-(2m+1)}}{H_{2m+1}^{-2m}} + \frac{N_{2m-1}^{-(2m-1)}}{H_{2m-1}^{-(2m-2)}} \quad \text{in case } \mathrm{L}_7 \tag{5.14}$$

$$w_{2m} = -\frac{N_{2m+1}^{-(2m+1)}}{H_{2m+1}^{-2m}} + \frac{N_{2m-1}^{-(2m-1)}}{H_{2m-1}^{-(2m-2)}} + \frac{H_{2m}^{-2m} H_{2m-2}^{-(2m-3)}}{H_{2m-1}^{-(2m-2)} H_{2m-1}^{-(2m-1)}} \quad \text{in case } \mathrm{L}_8. \tag{5.15}$$

Proof. (*Case* L_3). Comparison of coefficients for z^{m-1} in the recursion formula $B_{2m}(z) = ((v_{2m}/v_{2m-2})\, z^{-1} + w_{2m-1} + z)\, B_{2m-2}(z) + q_{2m} B_{2m-4}(z)$ gives $w_{2m-1} = b_{2m,m-1} - b_{2m-2,m-2}$, and the result follows from (5.1).

(*Case* L_4). Comparison of coefficients for z^{m-1} in the recursion formula $B_{2m}(z) = ((v_{2m}/v_{2m-2})\, z^{-1} + w_{2m-1} + z)\, B_{2m-2}(z) + q_{2m} z B_{2m-3}(z)$ gives $w_{2m-1} = b_{2m,m-1} - b_{2m-2,m-2} - q_{2m} b_{2m-3,m-2}$, and the result follows from (5.1) and (5.6) (Theorem 5).

(*Case* L_7). Comparison of coefficients for z^{-m} in the recursion formula $B_{2m+1}(z)=(z^{-1}+w_{2m}+(v_{2m+1}/v_{2m-1})z)\,B_{2m-1}(z)+q_{2m+1}B_{2m-3}(z)$ gives $w_{2m}=b_{2m+1,m}-b_{2m-1,-(m-1)}$, and the result follows from (5.1).

(*Case* L_8). Comparison of coefficients for z^{-m} in the recursion formula $B_{2m+1}(z) = (z^{-1}+w_{2m}+(v_{2m+1}/v_{2m-1})z)\,B_{2m-1}(z) + q_{2m+1}z^{-1}B_{2m-2}(z)$ gives $w_{2m}=b_{2m+1,-m}-b_{2m-1,-(m-1)}-q_{2m+1}b_{2m-2,-(m-1)}$, and the result follows from (5.1) and (5.10) (Theorem 5). ∎

6. Laurent Fractions Corresponding to Given Laurent Series

We have shown that to every Laurent fraction there corresponds a unique Laurent series $\sum_{p=0}^{\infty} c_p z^{-p}+\sum_{p=1}^{\infty} -c_{-p}z^p$, and this series satisfies the conditions $H_{2m}^{-2m}\neq 0$, $H_{2m+1}^{-2m}\neq 0$, $m=0, 1, 2, \ldots$. A series which satisfies these conditions is called *definite*. We shall now show that to every definite Laurent series there corresponds a Laurent fraction.

THEOREM 8. *Let a definite Laurent series* $\sum_{p=0}^{\infty} c_p z^{-p}+\sum_{p=1}^{\infty} -c_{-p}z^{-p}$ *be given. Then there exists a unique Laurent fraction* $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ *corresponding to the series. An index* $2m$ *is singular iff* $H_{2m}^{-(2m-1)}=0$, *and an index* $2m+1$ *is singular iff* $H_{2m+1}^{-(2m+1)}=0$. *The elements* $\alpha_k(z)$, $\beta_k(z)$ *of the fraction are fiven by the formulas of Theorems* 5–7.

Proof. The uniqueness of the Laurent fraction corresponding to a given Laurent series follows from Theorem 5–7.

We define elements $\alpha_k(z)$, $\beta_k(z)$ by the formulas of Theorems 5–7. The Laurent fraction $\mathrm{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ obtained in this way corresponds to a series $\sum_{p=0}^{\infty}\gamma_p z^{-p}+\sum_{p=1}^{\infty} -\gamma_{-p}z^p$. Then the elements $\alpha_k(z)$, $\beta_k(z)$ are given by the formulas of Theorems 5–7, where the Hankel determinants are constructed from the sequence $\{\gamma_p\colon p=0, \pm 1, \pm 2, \ldots\}$. It is readily verified that the system of Hankel determinants H_{2m}^{-2m}, H_{2m+1}^{-2m}, $H_{2m+1}^{-(2m+1)}$, $H_{2m}^{-(2m-1)}$ determines uniquely the sequence $\{c_p\colon p=0, \pm 1, \pm 2, \ldots\}$ from which it is constructed. Hence $\gamma_p=c_p$, $p=0, \pm 1, \pm 2, \ldots$, and the result follows. ∎

7. Correspondence between Contractive Laurent Fractions and Positive Definite Laurent Series

We shall call the sequence $\{c_p\colon p=0, \pm 1, \pm 2, \ldots\}$ *positive definite* if $H_{2m}^{-2m}>0$, $H_{2m+1}^{-2m}>0$ for all $m=0, 1, 2, \ldots$.

THEOREM 9. *Let the Laurent fraction* $\mathsf{K}_{k=1}^{\infty}(\alpha_k(z)/\beta_k(z))$ *correspond to the Laurent series* $\sum_{p=0}^{\infty} c_p z^{-p} + \sum_{p=1}^{\infty} -c_{-p} z^p$. *Then the fraction is contractive if and only if the sequence* $\{c_p\colon p=0, \pm 1, \pm 2, ...\}$ *is positive definite.*

Proof. Assume that the sequence $\{c_p\colon p=0, \pm 1, \pm 2, ...\}$ is positive definite. If $2m$ is singular then $(H_{2m}^{-2m})^2 = -H_{2m+1}^{-(2m+1)} H_{2m-1}^{-(2m-1)}$ by (4.4), hence $H_{2m+1}^{-(2m+1)} H_{2m-1}^{-(2m-1)} < 0$ and so $v_{2m-1} \cdot v_{2m+1} < 0$. Similarly if $2m+1$ is singular then $(H_{2m+1}^{-2m})^2 = -H_{2m+2}^{-(2m+1)} H_{2m}^{-(2m-1)}$ by (4.5), hence $H_{2m+2}^{-(2m+1)} H_{2m}^{-(2m-1)} < 0$, and so $v_{2m} \cdot v_{2m+2} < 0$. Thus C_0 is satisfied. Furthermore the following relations are seen to hold (in the cases L_1–L_8):

$$q_{2m} v_{2m} v_{2m-1} = \frac{(H_{2m}^{-(2m-1)})^2 \, H_{2m-2}^{-(2m-2)}}{(H_{2m-1}^{-(2m-1)})^2 \, H_{2m}^{-2m}} > 0, \tag{7.1}$$

$$q_{2m} v_{2m} = -\frac{(H_{2m}^{-(2m-1)})^2 \, H_{2m-3}^{-(2m-4)}}{H_{2m-1}^{-(2m-2)} H_{2m-2}^{-(2m-2)} H_{2m}^{-2m}} < 0, \tag{7.2}$$

$$q_{2m} = -\frac{H_{2m-1}^{-(2m-2)} H_{2m-4}^{-(2m-4)}}{H_{2m-2}^{-(2m-2)} H_{2m-3}^{-(2m-4)}} < 0, \tag{7.3}$$

$$q_{2m} v_{2m-2} = \frac{H_{2m-1}^{-(2m-2)} H_{2m-3}^{-(2m-4)}}{(H_{2m-2}^{-(2m-2)})^2} > 0, \tag{7.4}$$

hence $q_{2m} v_{2m} < 0$,

$$q_{2m+1} v_{2m+1} v_{2m} = \frac{(H_{2m+1}^{-(2m+1)})^2 \, H_{2m-1}^{-(2m-2)}}{(H_{2m}^{-2m})^2 \, H_{2m+1}^{-2m}} > 0, \tag{7.5}$$

$$q_{2m+1} v_{2m+1} = -\frac{(H_{2m+1}^{-(2m+1)})^2 \, H_{2m-2}^{-(2m-2)}}{H_{2m}^{-2m} H_{2m-1}^{-(2m-2)} H_{2m+1}^{-2m}} < 0, \tag{7.6}$$

$$q_{2m+1} = -\frac{H_{2m}^{-2m} H_{2m-3}^{-(2m-4)}}{H_{2m-1}^{-(2m-2)} H_{2m-2}^{-(2m-2)}} < 0, \tag{7.7}$$

$$q_{2m+1} v_{2m-1} = \frac{H_{2m}^{-2m} H_{2m-2}^{-(2m-2)}}{(H_{2m-1}^{-(2m-2)})^2} > 0, \tag{7.8}$$

hence $q_{2m+1} v_{2m+1} < 0$.

Thus also the conditions C_1–C_8 are satisfied, and the Laurent fraction is contractive.

Next assume that the Laurent fraction is contractive. We note that $H_0^0 > 0$, that $H_1^0 > 0$ if $n_1 = 1$, and that $H_1^0 > 0$, $H_2^2 > 0$ if $n_1 = 2$. Let k be given, and assume that $H_0^0 > 0, ... \, H_{2m-3}^{-(2m-4)} > 0$, $H_{2m-2}^{-(2m-2)} > 0$ if $n_{k-1} = 2m-2$, and that $H_0^0 > 0, ..., H_{2m-2}^{-(2m-2)} > 0$, $H_{2m-1}^{-(2m-2)} > 0$ if $n_{k-1} = 2m-1$.

(1′) Let $n_k = 2m$, $n_{k-1} = 2m-1$, $n_{k-2} = 2m-2$. Then by (7.1) and the assumption we get $H_{2m}^{-2m} > 0$.

(2′) Let $n_k = 2m$, $n_{k-1} = 2m-1$, $n_{k-2} = 2m-3$. Then by (7.2) and the assumption we get $H_{2m}^{-2m} > 0$.

(3′) Let $n_k = 2m$, $n_{k-1} = 2m-2$, $n_{k-2} = 2m-4$. Then by (7.3) and the assumption we get $H_{2m-1}^{-(2m-2)} < 0$. From (4.5) it follows that $H_{2m-2}^{-(2m-3)} H_{2m}^{-(2m-1)} < 0$, since $H_{2m-1}^{-(2m-1)} = 0$. This together with $(H_{2m-2}^{-(2m-3)} H_{2m}^{-(2m-1)})/(H_{2m-2}^{-(2m-2)} H_{2m}^{-2m}) = v_{2m} \cdot v_{2m-2} < 0$ and the assumption implies that $H_{2m}^{-2m} > 0$.

(4′) Let $n_k = 2m$, $n_{k-1} = 2m-2$, $n_{k-2} = 2m-3$. Then by (7.4) and the assumption we get $H_{2m}^{-(2m-2)} > 0$. As under (3′) we conclude that $H_{2m}^{-2m} > 0$.

(5′) Let $n_k = 2m+1$, $n_{k-1} = 2m$, $n_{k-1} = 2m-1$. Then by (7.5) and the assumption we get $H_{2m+1}^{-2m} > 0$.

(6′) Let $n_k = 2m+1$, $n_{k-1} = 2m$, $n_{k-2} = 2m-2$. Then by (7.6) and the assumption we get $H_{2m+1}^{-2m} > 0$.

(7′) Let $n_k = 2m+1$, $n_{k-1} = 2m-1$, $n_{k-2} = 2m-3$. Then by (7.7) and the assumption we get $H_{2m}^{-2m} > 0$. From (4.4) it follows that $H_{2m+1}^{-(2m+1)} H_{2m-1}^{-(2m-1)} > 0$, since $H_{2m}^{-(2m-1)} = 0$. This together with $(H_{2m+1}^{-(2m+1)} H_{2m-1}^{-(2m-1)})/(H_{2m+1}^{-2m} H_{2m-1}^{-(2m-2)}) = v_{2m+1} \cdot v_{2m-1} < 0$ and the assumption implies that $H_{2m+1}^{-2m} > 0$.

(8′) Let $n_k = 2m+1$, $n_{k-1} = 2m-1$, $n_{k-2} = 2m-2$. Then by (7.8) and the assumption we get $H_{2m}^{-2m} > 0$. As under (7′) we conclude that $H_{2m+1}^{-2m} > 0$.

It now follows by induction that the sequence is positive definite. ∎

8. Main Result

Our main results can be collected in the following theorem.

MAIN THEOREM. *Correspondence as defined in Section* 3 *introduces a one-to-one mapping between all Laurent fractions and all definite Laurent series. The contractive Laurent fractions are mapped exactly onto all positive definite Laurent series.*

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